

Heijal's theorem for parabolic projective structures on non compact surfaces.

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Abstract

We prove that the holonomy map from the set of equivalence classes of parabolic complex projective structures on non compact surfaces to the set of equivalence classes of parabolic representations of the fundamental group of the surface to $PSL_2(\mathbb{C})$ is a local biholomorphism.

1 Introduction.

Complex projective structure, monodromy map. Let S be an oriented surface. A *complex projective structure* on S is a $(PSL_2(\mathbb{C}), \mathbb{CP}^1)$ -structure on S , i.e. the data of a maximal atlas of charts mapping open sets of S into \mathbb{CP}^1 such that the transition functions are restrictions of Möbius transformations. Two complex projective structures σ_1 and σ_2 are equivalent if there exists an orientation-preserving diffeomorphism of S homotopic to the identity that pulls back the projective charts of σ_2 to projective charts of σ_1 . We denote by $\mathcal{P}(S)$ the classes of complex projective structures for the previous equivalence relation.

As Möbius transformations are holomorphic, a complex projective structure on S determines a complex structure on S . Hence, there is a natural map, sometimes called *forgetful map*:

$$\pi : \mathcal{P}(S) \longrightarrow \mathcal{T}(S),$$

where $\mathcal{T}(S)$ is the Teichmüller space of S .

Given a complex projective structure on S , we can define a pair (\mathcal{D}, ρ) where $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ is a morphism called *holonomy representation* which globalizes the transition functions and $\mathcal{D} : \tilde{S} \rightarrow \mathbb{CP}^1$ is a ρ -equivariant local homeomorphism¹ called *developing map* which globalizes the projective charts. The map \mathcal{D} is well defined up to post-composition by a Möbius transformation and the representation ρ is well defined up to conjugacy by this Möbius transformation. Hence, we have a natural map, called *holonomy map*:

$$Hol : \mathcal{P}(S) \longrightarrow \mathcal{R}(S)$$

1. \mathcal{D} is holomorphic for the underlying complex structure on S

where $\mathcal{R}(S)$ is the set of equivalence classes of morphisms $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ modulo the action by conjugacy.

The compact case. When S is the compact oriented surface S_g with genus $g \geq 2$, there are many results about the behaviour of the map Hol :

Theorem 1.1 (Poincaré). *In restriction to a fiber of the forgetful map π , the map Hol is injective.*

The spaces $\mathcal{P}(S)$ and $\mathcal{R}(S)$ can be given the structure of a complex manifold (see preliminaries). Heijal [He] proved that Hol is a local homeomorphism. It was proved by Earle [Ea] and Hubbard [Hu] that Hol is holomorphic. Hence, we have:

Theorem 1.2 (Heijal, Earle, Hubbard). *The map Hol is a local biholomorphism.*

The holonomy map is not surjective. But the celebrated theorem of Gallo-Kapovic-Marden [GKM] characterizes exactly the image of the holonomy map:

Theorem 1.3 (Gallo-Kapovic-Marden). *$Hol(\mathcal{P}(S_g))$ is exactly the classes of representations ρ satisfying:*

- ρ is non elementary (i.e. there is no Borel probability measure on CP^1 invariant by the action of the holonomy group $\rho(\pi_1(S))$).
- ρ can be lifted to a representation $\tilde{\rho} : \pi_1(S) \rightarrow SL(2, \mathbb{C})$.

The holonomy map is not globally injective: given a projective structure, there is a surgery called $(2\pi-)$ grafting, which produces a new projective structure without changing the holonomy representation. The following theorems of Goldman [Go] and Baba [Ba] say more or less that grafting is the only way to do this:

Theorem 1.4. – *Goldman: Every complex projective structure with quasi-Fuchsian holonomy representation can be obtained by grafting the uniformizing projective structure.*
– *Baba: Given two projective structures with the same purely loxodromic holonomy representation, we can pass from one to the other by grafting and ungrafting.*

The parabolic case. In this paper, we study the following natural question: if instead of considering the compact surface with genus $g \geq 2$, we consider the compact surface with genus g and n points deleted (with $3g - 3 + n > 0$), are the theorems of the previous part still valid? In this general context, the answer is no! For example, the space of complex projective structure $\mathcal{P}(S_{g,n})$ on $S_{g,n}$ is infinite dimensional whereas the space of representations of $\pi_1(S_{g,n})$ in $PSL_2(\mathbb{C})$ modulo conjugacy is finite dimensional. We deduce immediatly that the analogous of Heijal's and Poincaré's theorems cannot be true. So we need to restrict our space $\mathcal{P}(S_{g,n})$ in order to hope something. It is the goal of the following definition:

Definition 1.5. *A complex projective structure on $S_{g,n}$ is said to be parabolic if for any puncture p , there is a local coordinate z around p and a Möbius transformation A such that $A \circ \mathcal{D}(z) = \frac{1}{2i\pi} \log z$.*

In the previous definition, the developing map has to be thought as a multivalued map $\mathcal{D} : S_{g,n} \rightarrow \mathbb{CP}^1$. The set of parabolic complex projective structures modulo the equivalence relation defined at the beginning of the introduction, is denoted by $\mathcal{P}_{para}(S_{g,n})$.

Note that for a parabolic complex projective structure, in the local coordinate z around a puncture p given by definition 1.5, we have $A \circ \mathcal{D}(e^{2i\pi} \cdot z) = A \circ \mathcal{D}(z) + 1$. Hence $\mathcal{D}(e^{2i\pi} \cdot z) = A^{-1} \circ T_1 \circ A \circ \mathcal{D}(z)$ where $T_1 : z \mapsto z + 1$ is the translation. This means that the image of an element of $\pi_1(S_{g,n})$ which makes one loop around a puncture is a parabolic element of $PSL_2(\mathbb{C})$ (i.e. conjugated to $z \mapsto z + 1$). Such a representation is called parabolic and the quotient of all such representations by the conjugacy action is denoted by $\mathcal{R}_{para}(S_{g,n})$. Hence we have a map:

$$Hol : \mathcal{P}_{para}(S_{g,n}) \longrightarrow \mathcal{R}_{para}(S_{g,n}).$$

The analogous of Poincaré's theorem has been proved by I. Kra [Kr], namely:

Theorem 1.6 (Kra). *In restriction to a fiber of the forgetful map π , the map Hol is injective.*

As in the compact case, the two spaces $\mathcal{P}_{para}(S_{g,n})$ and $\mathcal{R}_{para}(S_{g,n})$ can be endowed with a structure of complex manifolds (see section 2). The goal of this paper is to prove that the analogous of Heijl's theorem also holds:

Theorem A. *The holonomy map Hol is a local biholomorphism.*

Questions.

Question 1.7. *Is the analogous of Gallo-Kapovic-Marden theorem 1.3 true in this context? The fundamental group of $S_{g,n}$ (with $n > 0$) being a free group, any morphism $\rho : \pi_1(S_{g,n}) \rightarrow PSL_2(\mathbb{C})$ can be lifted to a morphism $\tilde{\rho} : \pi_1(S_{g,n}) \rightarrow SL(2, \mathbb{C})$. Hence the precise question is: given a non-elementary parabolic representation ρ , does there exist a parabolic complex projective structure with holonomy ρ ? See also [GKM, Problem 12.2.1] for a question in the same vein (but slightly different).*

Question 1.8. *Are the analogous of Goldman's and Baba's theorems 1.4 true in our context?*

2 Preliminaries.

Complex projective structures. Let S be an oriented surface. A *complex projective structure* on S is the data of a pair (\mathcal{D}, ρ) where $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ is a morphism and $\mathcal{D} : \tilde{S} \rightarrow \mathbb{CP}^1$ is a ρ -equivariant local homeomorphism. Two complex projective structures (\mathcal{D}_1, ρ_1) and (\mathcal{D}_2, ρ_2) are said to be equivalent if there exists a $\pi_1(S)$ -equivariant homeomorphism $\tilde{\phi}$ of \tilde{S}^2 and a Möbius transformation A such that:

$$- \mathcal{D}_2 = A \circ \mathcal{D}_1 \circ \tilde{\phi}$$

$$2. \tilde{\phi} \circ \gamma = \gamma \circ \tilde{\phi}, \forall \gamma \in \pi_1(S)$$

$$- \rho_2 = A \circ \rho_1 \circ A^{-1}$$

The set of equivalence classes of complex projective structures on S is denoted by $\mathcal{P}(S)$. It is a classical fact that this definition is equivalent to the one given in the introduction.

Parabolic type projective structure. Let $S_{g,n}$ be the topological surface obtained from the oriented compact surface of genus g by deleting n points. Fix a universal covering map: $proj : \widetilde{S}_{g,n} \rightarrow S_{g,n}$. Let (\mathcal{D}, ρ) be a complex projective structure on $S_{g,n}$. Thinking \mathcal{D} as a multivalued map, (\mathcal{D}, ρ) is said to be *parabolic* at a puncture p if there is a neighborhood V of p , a homeomorphism $\phi : V \rightarrow \mathbb{D}^*$ and a Möbius transformation A such that:

$$A \circ \mathcal{D} \circ \phi^{-1}(z) = \frac{1}{2i\pi} \log z.$$

Equivalently, (\mathcal{D}, ρ) is parabolic at p if there exists:

- a neighborhood $V = C / \langle \gamma \rangle$ of p , where C is a connected component of $proj^{-1}(V)$ and $\gamma \in \pi_1(S_{g,n})$,
- a homeomorphism $\tilde{\phi} : C \rightarrow \mathbb{H}$ with $\tilde{\phi} \circ \gamma = T_1 \circ \tilde{\phi}$ where $T_1 : z \mapsto z + 1$ is the translation.
- a Möbius transformation A

satisfying

$$A \circ \mathcal{D} \circ \tilde{\phi}^{-1} = \iota : \mathbb{H} \hookrightarrow \mathbb{CP}^1.$$

A complex projective structure on $S_{g,n}$ is said to be *parabolic* if it is parabolic at all the punctures. The set of parabolic type projective structures on $S_{g,n}$ is denoted by $\mathcal{P}_{para}(S_{g,n})$.

Complex structures on the spaces $\mathcal{P}_{para}(S_{g,n})$ and $\mathcal{R}_{para}^{ne}(S_{g,n})$ and computation of their dimensions.

Complex structure and dimension of $\mathcal{P}_{para}(S_{g,n})$? For a more detailed explanation of the complex structure, we refer the reader to [Du, section 3]. Let us fix a complex structure X on $S_{g,n}$, i.e. $X \in \mathcal{T}(S_{g,n})$. The set of equivalent classes of complex projective structures on $S_{g,n}$ whose underlying complex structure is X is parameterized by the set of holomorphic quadratic differentials on X . To see this, write $X = \mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $PSL_2(\mathbb{R})$. If (\mathcal{D}, ρ) is a complex projective structure on $S_{g,n}$ whose underlying complex structure is X , then $\mathcal{D} : \mathbb{H} \rightarrow \mathbb{CP}^1$ is local biholomorphism. Hence, if $S_{\mathcal{D}} := \left(\frac{\mathcal{D}''}{\mathcal{D}'}\right)' - \frac{1}{2} \left(\frac{\mathcal{D}''}{\mathcal{D}'}\right)^2$ is the Schwarzian derivative of \mathcal{D} , by the ρ -equivariance of \mathcal{D} , the quadratic differential $S_{\mathcal{D}}(z) \cdot dz^2$ is invariant by the action of Γ and then descends to a quadratic differential Φ on X . Moreover, it can be proved (see [HPSG, Theorem 9.1.1]) that the projective structure is parabolic at a puncture p if and only if, in one (and hence in all) holomorphic coordinate sending p to 0, Φ has the following Laurent's series expansion:

$$\Phi = \left(\frac{1}{2z^2} + \frac{a_{-1}}{z} + a_0 + \dots \right) dz^2.$$

Hence if $X \in \mathcal{T}(S_{g,n})$, the fiber $\pi^{-1}(X)$ of the forgetful map $\pi : \mathcal{P}(S_{g,n}) \rightarrow \mathcal{T}(S_{g,n})$ is a complex affine space directed by the vectorial space of quadratic meromorphic differential on \overline{X} (the compactification of X) with poles of order at most 1 at the n punctures and no other poles. By Riemann-Roch theorem, this set has complex dimension $3g - 3 + n$ (see [DD, paragraph 6.1]). As $\mathcal{T}(S_{g,n})$ is also a complex manifold with complex dimension $3g - 3 + n$, we deduce that $\mathcal{P}(S_{g,n})$ is a complex manifold with complex dimension $6g - 6 + 2n$. There is also a natural topology on $\mathcal{P}(S_{g,n})$: we give the set of couples of maps (\mathcal{D}, ρ) the compact-open topology, and $\mathcal{P}(S_{g,n})$ inherits a quotient topology. As it is mentioned in [Du, section 3.3], note that this topology is the topology coming from the complex structure previously defined.

Complex structure and dimension of $\mathcal{R}_{para}^{ne}(S_{g,n})$? $\pi_1(S_{g,n})$ is a free group with $2g + n - 1$ generators (2 generators a_i and b_i for each of the g handles and $n - 1$ generators α_i corresponding to $n - 1$ punctures choosen among the n punctures) and $PSL_2(\mathbb{C}) \sim SO_3(\mathbb{C}) \subset \mathbb{C}^9$. Hence, the set $Hom(\pi_1(S_{g,n}), PSL_2(\mathbb{C}))$ has the structure of an affine complex algebraic variety given by the identification:

$$Hom(\pi_1(S_{g,n}), PSL_2(\mathbb{C})) \longrightarrow PSL_2(\mathbb{C})^{2g+n-1}$$

$$\rho \longmapsto (\rho(a_1), \dots, \rho(b_g), \rho(\alpha_1), \dots, \rho(\alpha_{n-1}))$$

This variety has complex dimension $3 \cdot (2g + n - 1)$. Define $Hom_{para}(\pi_1(S_{g,n}), PSL_2(\mathbb{C}))$ as the set of representations ρ such that $\rho(\alpha_i)$ is a parabolic matrix for $i \in \{1, \dots, n\}$. This algebraic subset is given by n algebraic equations ($Tr^2(\rho(\alpha_i)) = 4$), hence it has dimension $3 \cdot (2g + n - 1) - n = 6g + 2n - 3$. Now $PSL_2(\mathbb{C})$ acts algebraically on $Hom_{para}(\pi_1(S_{g,n}), PSL_2(\mathbb{C}))$. Hence the quotient $\mathcal{R}_{para}(S_{g,n})$ has 3 dimensions less, i.e. it has dimension $3g + 2n - 6$. It is classical that the quotient $\mathcal{R}_{para}(S_{g,n})$ is not Hausdorff but the subset $\mathcal{R}_{para}^{ne}(S_{g,n})$ (consisting of these equivalence classes of representations which are non-elementary) is Hausdorff, hence is a complex manifold.

Note that the holonomy of a parabolic complex projective structure is always non-elementary (see [CDFG, Lemma 10]). Hence we have the well defined map:

$$Hol : \mathcal{P}_{para}(S_{g,n}) \longrightarrow \mathcal{R}_{para}^{ne}(S_{g,n})$$

between complex manifolds of dimension $6g - 6 + 2n$. According to [Ea], the map Hol is holomorphic (see also [Hu]).

Suspension. Let S be a smooth surface and $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ be a group morphism. There is a classical construction which associates to ρ a \mathbb{CP}^1 -fiber bundle over S and a horizontal foliation. The construction is the following: $\pi_1(S)$ acts on the product $\tilde{S} \times \mathbb{CP}^1$ by $\gamma \cdot (\tilde{x}, z) = (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot z)$. This action is free and properly discontinuous. Hence the quotient $M_\rho = \tilde{S} \times \mathbb{CP}^1 / \pi_1(S)$ is a manifold which fibers over S via the projection $\Pi_\rho : [\tilde{x}, z] \rightarrow [\tilde{x}]$ and whose fibers are copy of \mathbb{CP}^1 . The horizontal foliation on the product $\tilde{S} \times \mathbb{CP}^1$ whose leaves are the sets $\tilde{S} \times \{z\}$ is globally preserved by the action of $\pi_1(S)$. Then, it descends to a foliation \mathcal{F}_ρ on M_ρ transverse to the fibers.

If there exists a smooth section $\sigma : S \rightarrow M_\rho$ of Π_ρ transverse to the foliation, then S is naturally endowed with a complex projective structure in the following way: the unique projective structure on any \mathbb{CP}^1 -fiber can be transported to $\sigma(S)$ by sliding along the leaves. This defines a complex projective structure on $\sigma(S)$ and thus on its Π_ρ projection S .

Reciprocally, if ρ is the holonomy representation associated to a complex projective structure (\mathcal{D}, ρ) on S , then the developing map \mathcal{D} gives a section σ of the fiber bundle associated with ρ . This section is defined by:

$$\sigma([\tilde{x}]) = [\tilde{x}, \mathcal{D}(\tilde{x})]$$

As \mathcal{D} is a local homeomorphism, this section is transverse to the horizontal foliation. Recall that a complex projective structure on S naturally induces a complex structure on S . Taking the product with the complex structure of the \mathbb{CP}^1 -fibers, we also get a complex structure on M_ρ . And the section $\sigma : S \rightarrow M_\rho$ is holomorphic (for the two complex structures which have just been defined on S and M_ρ).

In the sequel, we will denote by $(M_\rho, \Pi_\rho, \mathcal{F}_\rho)$ the foliation associated with a representation $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ and we will denote by $(M_\rho, \Pi_\rho, \mathcal{F}_\rho, \sigma)$ the pair “foliation/section” associated with a complex projective structure (\mathcal{D}, ρ) on S .

Compactification. Let $(M_\rho, \Pi_\rho, \mathcal{F}_\rho, \sigma)$ be the pair “foliation/section” associated with a parabolic projective structure (\mathcal{D}, ρ) on $S_{g,n}$. As $S_{g,n}$ is not compact, the total space M_ρ and the image of the section σ are not compact. Following [CDFG] or [DD], we are going to explain how we can compactify the total space, the section and the foliation. The new foliation will have singularities. Recall that for any puncture, there is a neighborhood of this puncture $V = C / \langle \gamma \rangle$, a homeomorphism $\tilde{\phi} : C \rightarrow \mathbb{H}$ satisfying $\tilde{\phi} \circ \gamma = T_1 \circ \tilde{\phi}$ and a Möbius transformation A such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\phi}} & \mathbb{H} \\ \downarrow \mathcal{D} & & \downarrow \iota \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

Note that this implies $\rho(\gamma) = A^{-1} \circ T_1 \circ A$. Indeed:

$$\rho(\gamma) \circ \mathcal{D} = \mathcal{D} \circ \gamma = A^{-1} \circ \tilde{\phi} \circ \gamma = A^{-1} \circ T_1 \circ \tilde{\phi} = A^{-1} \circ T_1 \circ A \circ \mathcal{D}.$$

Then, the following diagram is also commutative:

$$\begin{array}{ccc} C \times \mathbb{CP}^1 / \langle \gamma \rangle & \xrightarrow{[\tilde{\phi}, A]} & \mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1) \\ \downarrow \Pi_\rho & & \downarrow p_1 \\ C / \langle \gamma \rangle & \xrightarrow{\phi = [\tilde{\phi}]} & \mathbb{H} / \tau \sim \tau + 1 \end{array}$$

Moreover, it can be easily checked that $\Phi := [\tilde{\phi}, A]$ sends \mathcal{F}_ρ on the horizontal foliation of $\mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1)$ and that $\Phi \circ \sigma = s_0 \circ \phi$ where s_0 is the diagonal

section: $s_0 := [\iota, \iota] : \mathbb{H} / \tau \sim \tau + 1 \longrightarrow \mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1)$.

We are going to glue the horizontal foliation of the manifold

$$\mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1) \sim \mathbb{D}^* \times \mathbb{CP}^1$$

with a foliation of $\mathbb{D} \times \mathbb{CP}^1$ using the following local model: consider the equation

$$\frac{dv}{du} = -\frac{1}{2i\pi u}, (u, v) \in \mathbb{D} \times \mathbb{CP}^1 \quad (1)$$

The solutions of this equation define a foliation in $\mathbb{D} \times \mathbb{CP}^1$ which is transverse to every fibre $\{u\} \times \mathbb{CP}^1$ except the fibre $\{0\} \times \mathbb{CP}^1$ which is an invariant projective line with only one singularity at $(0, \infty)$. The leaf of the foliation through $(u_0, v_0) \in \mathbb{D}^* \times \mathbb{CP}^1$ is the graph of the multivalued map:

$$v_{(u_0, v_0)}(u) = -\frac{1}{2i\pi} \log \left(\frac{u}{u_0} \right) + v_0$$

The map

$$\begin{aligned} \mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1) &\longrightarrow \mathbb{D}^* \times \mathbb{CP}^1 \\ (\tau, z) &\longmapsto (e^{2i\pi\tau}, z - \tau) \end{aligned}$$

is a biholomorphism which preserves the fibres and sends the horizontal foliation to the foliation defined by equation (1). So we can glue these foliations together. Moreover, the diagonal section s_0 is sent to the constant section equal to 0.

Doing this for any puncture, one gets a singular foliation $\overline{\mathcal{F}}$ (with real dimension 2) on a compact manifold \overline{M}_ρ (with real dimension 4) which is a \mathbb{CP}^1 -fiber bundle over the compact surface $S_g = \overline{S}_{g,n}$. The foliation is transverse to all the fibres except a finite number which are invariant projective complex lines with one singularity. Moreover, the section $\sigma : S_{g,n} \rightarrow M_\rho$ associated with the developing map can be compactified as a section $\overline{\sigma} : S_g \rightarrow \overline{M}_\rho$ transverse to $\overline{\mathcal{F}}$ and which does not pass through the singularities.

Remark 2.1. Take any other smooth section $\overline{\sigma}' : S_g \rightarrow \overline{M}_\rho$ which is transverse to the foliation $\overline{\mathcal{F}}$ and which does not pass through the singularities. The restriction $\sigma' := \overline{\sigma}'|_{S_{g,n}}$ defines a complex projective structure on $S_{g,n}$. This (new) complex projective structure is still parabolic. Indeed, for any puncture p , the vertical leaf of $\overline{\mathcal{F}}$ above p intersects transversely $\overline{S} = \overline{\sigma}(S_g)$ and $\overline{S}' = \overline{\sigma}'(S_g)$ in two points a and a' . Then, flowing along the leaves naturally defines a diffeomorphism h going from a neighborhood of a in \overline{S} to a neighborhood of a' in \overline{S}' . The homeomorphism $\Psi := \Pi_\rho \circ h \circ \sigma$ between two punctured neighborhood of p in $S_{g,n}$ can be lifted, at the level of the universal cover, in a $<\gamma>$ -equivariant homeomorphism $\tilde{\Psi}$ which satisfies by construction $\mathcal{D}' \circ \tilde{\Psi} = \mathcal{D}$ for some developing maps \mathcal{D} and \mathcal{D}' associated respectively to σ and σ' . Hence, from the definition of a parabolic projective structure (see section 2), if the projective structure associated with σ is parabolic, then the one associated with σ' is also parabolic.

3 *Hol* is locally injective.

As the quotient $\mathcal{R}_{para}^{ne}(S_{g,n})$ of non elementary representations modulo conjugacy is a Hausdorff complex manifold, it is enough to prove that if (\mathcal{D}_1, ρ) and (\mathcal{D}_2, ρ) are two parabolic complex projective structures on $S_{g,n}$ with the same holonomy ρ and close enough, then they are equivalent, i.e. there exists $\tilde{\phi}$ a $\pi_1(S_{g,n})$ -equivariant homeomorphism of $\widetilde{S_{g,n}}$ such that $\mathcal{D}_2 = \mathcal{D}_1 \circ \tilde{\phi}$.

(\mathcal{D}_1, ρ) and (\mathcal{D}_2, ρ) being of parabolic type, for any puncture p and for $i \in \{1, 2\}$, one can find neighborhoods $V_i = C_i / \langle \gamma \rangle$ of p and homeomorphisms $\tilde{\phi}_i : C_i \rightarrow \mathbb{H}$ satisfying $\tilde{\phi}_i \circ \gamma = T_1 \circ \tilde{\phi}_i$ such that the following diagram commutes:

$$\begin{array}{ccccc} C_1 & \xrightarrow{\tilde{\phi}_1} & \mathbb{H} & \xleftarrow{\tilde{\phi}_2} & C_2 \\ \downarrow \mathcal{D}_1 & & \downarrow \iota & & \downarrow \mathcal{D}_2 \\ \mathbb{CP}^1 & \xrightarrow{A_1} & \mathbb{CP}^1 & \xleftarrow{A_2} & \mathbb{CP}^1 \end{array}$$

Lemma 3.1. *For all $t \in \mathbb{R}$, denote by T_t the translation $z \mapsto z + t$. If we change the pair $(\tilde{\phi}_2, A_2)$ in the previous diagram by $(\tilde{\phi}_2^t, A_2^t) := (T_t \circ \tilde{\phi}_2, T_t \circ A_2)$, we do not affect anything, i.e.:*

1. *the diagram is still commutative: $A_2^t \circ \mathcal{D}_2 \circ (\tilde{\phi}_2^t)^{-1} = \iota$*
2. *$\tilde{\phi}_2^t \circ \gamma = T_1 \circ \tilde{\phi}_2^t$.*

Moreover, for one value of t , we have $A_2^t = A_1$.

Proof. Items 1 and 2 are trivial. In order to prove the last assertion of the lemma, we denote $A^t := A_1^{-1} \circ A_2^t$. Then, $A^t = (A_1^{-1} \circ A_2) \circ (A_2^{-1} \circ T_t \circ A_2)$. The matrix $A_1^{-1} \circ A_2$ commutes with $\rho(\gamma)$. Moreover $\rho(\gamma)$ is parabolic with $A_2^{-1}(\infty)$ as only fixed point. We deduce that $A_1^{-1} \circ A_2$ is either the identity or is parabolic with $A_2^{-1}(\infty)$ as only fixed point. Then for one value of t , we have: $A_2^{-1} \circ T_t \circ A_2 = (A_1^{-1} \circ A_2)^{-1}$. ■

Choosing the value of t satisfying the previous lemma, we obtain the following commutative diagram:

$$\begin{array}{ccccc} C_1 & \xrightarrow{\tilde{\phi}_1} & \mathbb{H} & \xleftarrow{\tilde{\phi}_2} & C_2 \\ & \searrow \mathcal{D}_1 & \downarrow \iota & \swarrow \mathcal{D}_2 & \\ & & \mathbb{CP}^1 & & \end{array}$$

The map $\tilde{\phi} : C_1 \rightarrow C_2$ defined by $\tilde{\phi} = \tilde{\phi}_2^{-1} \circ \tilde{\phi}_1$ is $\langle \gamma \rangle$ -equivariant and satisfies $\mathcal{D}_1 = \mathcal{D}_2 \circ \tilde{\phi}$. We can continue $\tilde{\phi}$ to all $proj^{-1}(V_1) = \bigsqcup_{\alpha \in \pi_1(S_{g,n})} \alpha \cdot C_1$ in the following way: for $x \in C_1$ and $\alpha \in \pi_1(S_{g,n})$, we define $\tilde{\phi}(\alpha \cdot x) = \alpha \cdot \tilde{\phi}(x)$. Then, we get a $\pi_1(S_{g,n})$ -equivariant homeomorphism $\tilde{\phi} : proj^{-1}(V_1) \rightarrow proj^{-1}(V_2)$ which still satisfies the equality $\mathcal{D}_1 = \mathcal{D}_2 \circ \tilde{\phi}^3$.

3. Lemma 3.1 was important to keep the equality satisfied

The previous construction can be done for any puncture p . We denote, for $i \in \{1, 2\}$, by $V_i(p) = C_i(p) / \langle \gamma \rangle$ the neighborhoods of the puncture p and $\tilde{\phi}_p : \text{proj}^{-1}(V_1(p)) \rightarrow \text{proj}^{-1}(V_2(p))$ the homeomorphisms as in the previous part. Let $V'_1(p) = C'_1(p) / \langle \gamma \rangle$ be a smaller neighborhood of p (i.e. $\overline{V'_1(p)} \subset V_1(p)$) and take $C'_2(p) = \tilde{\phi}_p(C'_1(p))$ and $V'_2(p) = \text{proj}(C'_2(p))$. Denote also by $K_i = S_{g,n} \setminus \bigsqcup_p V'_i(p)$ and by $\tilde{K}_i = \text{proj}^{-1}(K_i)$. We are going to prove that there is a $\pi_1(S_{g,n})$ -equivariant homeomorphism $\tilde{\psi} : \tilde{K}_1 \rightarrow \tilde{K}_2$ satisfying $\mathcal{D}_1 = \mathcal{D}_2 \circ \tilde{\psi}$.

For this, let $(M_\rho, \Pi_\rho, \mathcal{F}_\rho)$ be the foliation associated with the representation ρ . The developing maps \mathcal{D}_1 and \mathcal{D}_2 induce two sections σ_1 and σ_2 of the bundle. As the K_i are compact, if (\mathcal{D}_1, ρ) and (\mathcal{D}_2, ρ) are close enough, then there exists a holonomy map $h : \sigma_1(K_1) \rightarrow \sigma_2(K_2)$ close to the identity. Now, the map $\psi : K_1 \rightarrow K_2$ defined by $\psi = \Pi_\rho \circ h \circ \sigma_1$ can be lifted uniquely in a $\pi_1(S_{g,n})$ -equivariant homeomorphism $\tilde{\psi} : \tilde{K}_1 \rightarrow \tilde{K}_2$ satisfying $\mathcal{D}_1 = \mathcal{D}_2 \circ \tilde{\psi}$.

To conclude, we just have to prove that for all the punctures p , the maps $\tilde{\phi}_p$ and $\tilde{\psi}$ coincide on their intersection $\text{proj}^{-1}(V_1(p) \setminus V'_1(p))$. Let $x \in C_1(p) \setminus C'_1(p)$, we have:

$$\mathcal{D}_1(x) = \mathcal{D}_2(\tilde{\phi}_p(x)) = \mathcal{D}_2(\tilde{\psi}(x)).$$

As $\tilde{\phi}_p(x)$ and $\tilde{\psi}(x)$ belong to $C_2(p) \setminus C'_2(p)$ and \mathcal{D}_2 is injective on $C_2(p)$, we deduce that $\tilde{\phi}_p(x) = \tilde{\psi}(x)$. As the maps $\tilde{\phi}_p$ and $\tilde{\psi}$ are both $\pi_1(S_{g,n})$ -equivariant, we deduce that they coincide on $\text{proj}^{-1}(V_1(p) \setminus V'_1(p))$.

We have seen in section 2 that $\mathcal{P}_{para}(S_{g,n})$ and $\mathcal{R}_{para}^{ne}(S_{g,n})$ are two complex manifolds with the same dimension and that Hol is a holomorphic map. We deduce that Hol is open. We have just proved that Hol is locally injective. Hence Hol is a local biholomorphism and theorem A is proved. Nevertheless, in this last section, we are going to give a geometric proof of the fact that Hol is open without using the fact that it is holomorphic.

4 Hol is open.

Let (\mathcal{D}_0, ρ_0) be a parabolic complex projective structure on $S_{g,n}$. We have to prove that, if ρ is close enough to ρ_0 , then ρ is the holonomy representation of a parabolic complex projective structure.

In the case where the surface is compact, the proof is classical (see for example [LM, Theorem 1.3] or [Me, Proposition 1.7.1]), the idea being the following: for every perturbation ρ of ρ_0 , construct the suspended foliation $(M_\rho, \Pi_\rho, \mathcal{F}_\rho)$ associated to ρ . If ρ is close to ρ_0 , then $(M_\rho, \Pi_\rho, \mathcal{F}_\rho)$ is “close” to $(M_{\rho_0}, \Pi_{\rho_0}, \mathcal{F}_{\rho_0})$ and the section σ_0 of $(M_{\rho_0}, \Pi_{\rho_0}, \mathcal{F}_{\rho_0})$ associated to the developing map \mathcal{D}_0 (transverse to \mathcal{F}_0) can be transported to define a section σ of $(M_\rho, \Pi_\rho, \mathcal{F}_\rho)$ which is still transverse to \mathcal{F}_ρ . This section defines a developing map \mathcal{D} with holonomy ρ .

Trying to adapt this proof to our case, we are faced with two difficulties. Firstly, the base surface is not compact anymore. Hence the perturbed section σ could be not transverse to the foliation anymore even if the perturbation is arbitrarily small.

Secondly, the projective structure associated with the perturbed section could be not parabolic anymore. The idea of the proof is to compactify the bundles as explained in section 2.

More precisely, let U_{ρ_0} be a contractible neighborhood of ρ_0 in the set of all parabolic representations of $\pi_1(S_{g,n})$. The group $\pi_1(S_{g,n})$ acts on $\widetilde{S_{g,n}} \times \mathbb{CP}^1 \times U_{\rho_0}$ by:

$$\alpha \cdot (\tilde{x}, z, \rho) \longmapsto (\alpha \cdot \tilde{x}, \rho(\alpha) \cdot z, \rho).$$

The quotient is a \mathbb{CP}^1 -bundle E over $S_{g,n} \times U_{\rho_0}$. It is endowed with a foliation: the one whose leaves are the quotients of the sets $\widetilde{S_{g,n}} \times \{z\} \times \{\rho\}$. For every $\rho \in U_{\rho_0}$, the restriction E_ρ of E over $S_{g,n} \times \{\rho\}$ is the suspension of ρ .

As (\mathcal{D}_0, ρ_0) is parabolic, for any puncture, there is a neighborhood $C / \langle \gamma \rangle$, a homeomorphism $\tilde{\phi} : C \rightarrow \mathbb{H}$ satisfying $\tilde{\phi} \circ \gamma = T_1 \circ \tilde{\phi}$ and a Möbius transformation A_0 such that:

$$A_0 \circ \mathcal{D}_0 \circ \tilde{\phi}^{-1} = \iota : \mathbb{H} \hookrightarrow \mathbb{CP}^1.$$

For every $\gamma \in \pi_1(S_{g,n})$ which makes one loop around a puncture and for every $\rho \in U_{\rho_0}$, the Möbius transformation $\rho(\gamma)$ is parabolic. So, we can find an analytic family of Möbius transformations $\{A_\rho; \rho \in U_{\rho_0}\}$ satisfying $A_{\rho_0} = A_0$ and $A_\rho \circ \rho(\gamma) \circ A_\rho^{-1} = T_1$. Then we have the following commutative diagram:

$$\begin{array}{ccc} C \times \mathbb{CP}^1 \times U_{\rho_0} / \langle \gamma \rangle & \xrightarrow{[\tilde{\phi}, A_\rho, id]} & \mathbb{H} \times \mathbb{CP}^1 / (\tau, z) \sim (\tau + 1, z + 1) \times U_{\rho_0} \\ \downarrow & & \downarrow \\ C / \langle \gamma \rangle \times U_{\rho_0} & \xrightarrow{([\tilde{\phi}], id)} & \mathbb{H} / \tau \sim \tau + 1 \times U_{\rho_0} \end{array}$$

We can compactify using the local model explained in the preliminaries. We get a \mathbb{CP}^1 -fiber bundle \overline{E} over $S_g \times U_{\rho_0}$. By hypothesis, the compactified section $\overline{\sigma}_0$ over $S_g \times \{\rho_0\}$ is transverse to the foliation and does not meet any singularities. The open set U_{ρ_0} being contractible, by the universal property of the fibre bundles, $\overline{\sigma}_0$ can be extended to a global section of \overline{E} over $S_g \times U_{\rho_0}$. For ρ close enough to ρ_0 , the restriction $\overline{\sigma}_\rho$ of this global section to $S_g \times \{\rho\}$ is still transverse to the foliation and does not meet any singularities. Then, as explained in the preliminaries (see in particular remark 2.1), the restriction σ_ρ of $\overline{\sigma}_\rho$ to $S_{g,n} \times \{\rho\}$ defines a complex projective structure of parabolic type on $S_{g,n}$ with holonomy ρ .

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